THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH1010H University Mathematics 2017-2018

Suggested Solution to Assignment 4

1. (a) $\dfrac{dy}{dx} = \dfrac{\ln 2 \cdot 2^x + 5}{\ln 10 \cdot (2^x + 5x)}$.

   (b) $\dfrac{dy}{dx} = \cos x \cdot \ln x + \dfrac{1}{x} \sin x$.

   (c) $\dfrac{dy}{dx} = \dfrac{\sin x - x \cos x}{\sin^2 x}$.

   (d) $\dfrac{dy}{dx} = \dfrac{x \cos x - \sin x}{x^2}$

   (e) $\dfrac{dy}{dx} = \dfrac{(x^3 + 3x^2) \cdot e^x}{2\sqrt{x^3 e^x + 1}}$.

   (f) $\dfrac{dy}{dx} = -\dfrac{e^{\cot x}}{\sin^2 x}$.

   (g)
$$3 \ln y = 7x + 3 \ln(x^2 + 1) - 5 \ln(x + 1),$$
   then taking derivative, we have
$$3 \frac{\frac{dy}{dx}}{y} = 7 + \frac{6x}{x^2 + 1} - \frac{5}{x + 1}$$
$$\frac{dy}{dx} = 3 \sqrt[3]{\frac{e^{7x}(x^2 + 1)^3}{(x + 1)^5}} (7 + \frac{6x}{x^2 + 1} - \frac{5}{x + 1})$$

   (h) $\ln y = \cos x \ln x$, then $y'/y = -\ln x \cdot \sin x + (\cos x)/x$, $\dfrac{dy}{dx} = x^{\cos x}(-\ln x \cdot \sin x + \dfrac{\cos x}{x})$

   (i) $x = \tan y$. Then taking derivative respect to x, we get $1 = \frac{1}{\cos^2 y} \frac{dy}{dx}$. Notice $\tan^2 y = \frac{1 - \cos^2 y}{\cos^2 y} = x^2$, $\cos^2 y = \frac{1}{x^2 + 1}$. Therefore $\dfrac{dy}{dx} = \dfrac{1}{x^2 + 1}$.

2. Taking $x = 1, y = 0$, L.H.S $= 1 + 2 \cdot 0 + 0^2 = 1 =$ R.H.S, hence $(1, 0)$ is on the curve $\mathcal{C}$. Taking derivatives of $x^3 + 2xy + y^2 = 1$ with respect to x, we have $3x^2 + 2(y + xy') + 2yy' = 0$, hence
$$y' = -\frac{3x^2 + 2y}{2x + 2y}.$$
   The tangent at $(1, 0)$ is then $y'|_{(1,0)} = -\frac{3}{2}$, and the tangent line is therefore
$$y = -\frac{3}{2}(x - 1).$$

3. Proof. Taking derivatives of the equation $f(-x) = -f(x)$, using chain rule, we have L.H.S $= f'(-x) \cdot (-x)' = -f'(-x)$ and R.H.S $= -f'(x)$, therefore $f'(-x) = f(x)$.

4. (a) f(0)=1, and
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{1 + x + x^2 - 1}{x} = \lim_{x \to 0^+} 1 + x = 1;$$
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{1 + x - 1}{x} = \lim_{x \to 0^-} 1 = 1.$$
   Therefore $\lim_{x \to 0} \frac{f(x) - f(0)}{x} = 1$. Hence $f(x)$ is differential at $x = 1$ and $f'(0) = 1$.

(b)

$$f'(x) = \begin{cases} 1+2x, & \text{if} \quad x > 0, \\ 1, & \text{if} \quad x = 0, \\ 1, & \text{if} \quad x < 0. \end{cases}$$

(c) First notice $f'(0) = 1$.

$$\lim_{x\to 0^+} \frac{f'(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{1+2x-1}{x} = 2;$$
$$\lim_{x\to 0^-} \frac{f'(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{1-1}{x} = 0 \neq 2.$$

Therefore $f'(x)$ is not differentiable at $x = 0$.

5. (a)

$$\lim_{x\to 0} \frac{f(x) - 0}{x} = \lim_{x\to 0} x \sin(\frac{1}{x^2}) = 0.$$

(The last equality is by Sandwich theorem, since $0 \leq |x| \cdot |\sin(\frac{1}{x^2})| \leq |x|$.) Therefore $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

(b)

$$f'(x) = \begin{cases} 2x\sin(\frac{1}{x^2}) - 2\frac{1}{x}\cos\frac{1}{x^2}, & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0.. \end{cases}$$

(c) $f'(x)$ is not differentiable at $x = 0$. The reason is, consider the limit

$$h(x) := \frac{f'(x) - f'(0)}{x} = 2\sin\frac{1}{x^2} - 2\frac{1}{x^2}\cos\frac{1}{x^2}.$$

Consider the sequence

$$x_n = \frac{1}{\sqrt{2\pi n}}.$$

Then $x_n \to 0$, and

$$h(x_n) = \sin(2\pi n) - 2(2\pi n)\cos(2\pi n) = -4\pi n.$$

When $n \to \infty$, $h(x_n)$ does not have limit. Therefore $\lim_{x\to 0} h(x)$ does not exist. By definition, $f'(x)$ is not differentiable at $x = 0$.

Remark. Since a function $F(x)$ is differentiable at $x = 0$, implies $F(x)$ is continuous at $x = 0$. Hence one can also directly show $f'(x)$ is not even continuous at $x = 0$: $\lim_{x\to 0} f'(x)$ does not exist. The reason is, since the first $\lim_{x\to 0} 2x\sin(\frac{1}{x^2}) = 0$ by sandwich theorem, assume $\lim_{x\to 0} f'(0) = A$ for some real number $A \in \mathbb{R}$, then this would imply $\lim_{x\to 0} -2\frac{1}{x}\cos\frac{1}{x^2} = A - 2$ exists. But

$$g(x) := -2\frac{1}{x}\cos\frac{1}{x^2}$$

does not have a limit when $x \to 0$, for example, if we take

$$x_n = \frac{1}{\sqrt{2\pi n}}$$

Then $\lim_{n\to\infty} x_n$. But

$$g(x_n) = -2\sqrt{2\pi n}\cos(2\pi n) = -2\sqrt{2\pi n}$$

does not have a limit as $n \to \infty$ (it 'has limit $-\infty$'). Contradiction. Therefore $f'(x)$ is not continuous at $x = 0$, therefore is not differentiable at $x = 0$.

6. The equation given in this exercise is:

$$f(x+y) = f(x) + f(y) + f(x)f(y) \quad \text{for all} \quad x, y \in \mathbb{R} \tag{1}$$

Proof.

(a) (i). Take $y = 0$ in (1), we have

$$f(0)(1 + f(x)) = 0 \quad \text{for all} \quad x \in \mathbb{R} \tag{2}$$

(ii). Take x= 0 in equation (**??**) we have

$$f(0)(1 + f(0)) = 0.$$

Therefore $f(0) = 0$ or $f(0) = -1$. If $f(0) = -1$, then from equation (**??**) we have $f(x) = -1$ for all $x \in \mathbb{R}$. This contradicts with $f(x)$ is a *non-constant* function. Therefore $f(0) = 0$.

(iii). Suppose that $f(y_0) = -1$ for some $y_0 \in \mathbb{R}$, then from (**??**)

$$f(x) = f((x-y_0)+y_0) = f(x-y_0)+f(y_0)+f(x-y_0)f(y_0) = f(x-y_0)-1-f(x-y_0) = -1,$$

for all $x \in \mathbb{R}$. This again contradicts with $f(x)$ is a *non-constant* function.

(b)

$$f(x) = f(\frac{x}{2} + \frac{x}{2}) = f(\frac{x}{2}) + f(\frac{x}{2}) + f(\frac{x}{2})^2 = (1 + f(\frac{x}{2}))^2 - 1 \geq -1.$$

But since we already show, f can never have value -1, $f(\frac{x}{2}) \neq -1$, therefore $f(x) > -1$.

(c) Take $x = x, y = h$ in (**??**), we have

$$f(x+h) = f(x) + f(h) + f(x)f(h)$$

Therefore for any given $x \in \mathbb{R}$, we have

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = (f(x) + 1) \lim_{h \to 0} \frac{f(h)}{h} = a(f(x) + 1).$$

Hence $f(x)$ is differentiable, and $f'(x) = a(1 + f(x))$. If $a = 0$, then $f'(x) = 0$, $f(x)$ is a constant function, contradiction. Therefore $a \neq 0$.

(d) Since

$$(\ln(1 + f(x)))' = \frac{f'(x)}{1 + f(x)} = a.$$

Therefore $\ln(1 + f(x)) = ax + C$, for some constant C. But we know $f(0) = 0$, therefore $\ln(1 + 0) = a \cdot 0 + C$, we get $C = 0$. Hence $\ln(1 + f(x)) = ax$, and $f(x) = e^{ax} - 1$.

7. (a) Consider $f(x) = \ln(1 + x)$, which is differential function for $x > -1$. For $x > 0$, apply mean value theorem, there exists $c \in (0, x)$ such that

$$\frac{\ln(x+1) - \ln 1}{x} = f'(c),$$

i.e.

$$\frac{\ln(x+1)}{x} = \frac{1}{1+c}.$$

From $0 < c < x$ we get

$$\frac{1}{1+x} < \frac{\ln(x+1)}{x} < \frac{1}{1} = 1.$$

(b) Consider the $f(x) = x^n$, $(n > 1)$ which is differential function for $x > 0$, and let $0 < y < x$. Apply mean value theorem, there exists $c \in (y, x)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(c),$$

i.e.

$$\frac{x^n - y^n}{x - y} = nc^{n-1}.$$

Since $0 < y < c < x$, we obtain

$$ny^{n-1}(x - y) < x^n - y^n < nx^{n-1}(x - y).$$

8. Let $f(x) = \ln(1 + x) - \frac{2x}{2 + x} - \frac{x^3}{12}$. Then $f(0) = 0$, and when $x > 0$, we have

$$f'(x) = \frac{1}{1 + x} - \frac{4}{(x + 2)^2} - \frac{x^2}{4} = -\frac{x^3(x^2 + 5x + 8)}{4(x + 1)(x + 2)^2} < 0.$$

Then when $x > 0$, $f(x) < f(0) = 0$.

Remark. One can also see this from, when $x > 0$,

$$f'(x) = \frac{1}{x + 1} - \frac{4}{(x + 2)^2} - \frac{x^2}{4} = \frac{x^2}{(x + 1)(2 + x)^2} - \frac{x^2}{4} < 0.$$

9. $f(x) = \frac{(x + n + 1)^{n+1}}{(x + n)^n}$. Then when $x > 0$,

$$f'(x) = \frac{x(x + n + 1)^n}{(x + n)^{n+1}} > 0.$$

Therefore $f(x)$ is strictly increasing on $(0, +\infty)$. Notice $f(x) = (x+n)(1+\frac{1}{x + n})^{n+1}$ is continuous on $[0, +\infty)$, hence we have $\lim_{x\to 0^+} f(x) < f(1)$, i.e.

$$n(1 + \frac{1}{n})^{n+1} < (1 + n)(1 + \frac{1}{n + 1})^{n+1}.$$

But the left hand side is just $(1 + n)(1 + \frac{1}{n})^n$. Therefore we are done.

10. (a)

$$h'(x) = \frac{\ln x - 1}{(\ln x)^2}.$$

When $x = e$, $h'(x) = 0$; When $1 < x < e$, $h'(x) < 0$; When $x > e$, $h'(x) > 0$. Then $x = e$ is the minimum point of $h(x)$. Therefore $h(x) \geq h(e) = e$ for all $x > 1$.

(b)

$$f'(x) = b^x x^{b-1}(b - x \ln b).$$

When $x = \frac{b}{\ln b}$, $f'(x) = 0$; When $1 < x < \frac{b}{\ln b}$, $f'(x) > 0$, therefore $f(x)$ is strictly increasing on $(1, \frac{b}{\ln b})$; When $x > \frac{b}{\ln b}$, $f'(x) < 0$, therefore $f(x)$ is strictly decreasing on $(\frac{b}{\ln b}, +\infty)$.

(c) From (a) we know $e \leq \frac{b}{\ln b}$. Therefore for given a, b such that $1 < a < b < e$, then $1 < a < b < \frac{b}{\ln b}$. From (b) we have $f(a) < f(b)$, i.e.

$$\frac{a^b}{b^a} < \frac{b^b}{b^b} = 1.$$

Therefore $a^b < b^a$.